Noise corrections to stochastic trace formulas

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Abstract

We review studies of an evolution operator $\mathcal{L}$ for a discrete Langevin equation with a strongly hyperbolic classical dynamics and a Gaussian noise. The leading eigenvalue of $\mathcal{L}$ yields a physically measurable property of the dynamical system, the escape rate from the repeller. The spectrum of the evolution operator $\mathcal{L}$ in the weak noise limit can be computed in several ways. A method using a local matrix representation of the operator allows to push the corrections to the escape rate up to order eight in the noise expansion parameter. These corrections then appear to form a divergent series. Actu-

ally, via a cumulant expansion, they relate to analogous divergent series for other quantities, the traces of the evolution operators $\mathcal{L}^n$. Using an integral representation of the evolution operator $\mathcal{L}$, we then investigate the high order corrections to the latter traces. Their asymptotic behavior is found to be controlled by sub-dominant saddle points previously neglected in the perturbative expansion, and to be ultimately described by a kind of trace formula.

I. INTRODUCTION

In the statistical theory of dynamical systems the development of the densities of particles is governed by a corresponding evolution operator. For a repeller, the leading eigenvalue of this operator $\mathcal{L}$ yields a physically measurable property of the dynamical system, the escape rate from the repeller. In the case of deterministic flows, the periodic orbit theory [1,2] yields explicit and numerically efficient formulas for the spectrum of $\mathcal{L}$ as zeros of its spectral determinant [3].

Upon all dynamical evolutions in nature, stochastic processes of various strength have an influence. In a series of papers [4–7] the effects of noise on measurable properties such as dynamical averages in classical chaotic dynamical systems were systematically accounted. The theory developed is closely related to the semi-classical $\hbar$ expansions [8–10] based on Gutzwiller's formula for the trace in terms of classical periodic orbits [11], in that both are perturbative theories in the noise strength or $\hbar$, derived from saddle-point expansions of a path integral containing a dense set of unstable stationary points. The analogy with quantum mechanics and field theory was made explicit in [4] where Feynman diagrams were used to find the lowest nontrivial noise corrections to the escape rate.

An elegant method, inspired by the classical perturbation theory of celestial mechanics, was that of smooth conjugations [5]. In this approach the neighborhood of each saddle point was flattened by an appropriate coordinate transformation, so the focus shifted from the original dynamics to the properties of the transformations involved. The expressions

obtained for perturbative corrections in this approach were much simpler than those found from the equivalent Feynman diagrams. Using these techniques, we were able to extend the stochastic perturbation theory to the fourth order in the noise strength.

In [6] we developed a third approach, based on construction of an explicit matrix representation of the stochastic evolution operator. The numerical implementation required a truncation to finite dimensional matrices, and was less elegant than the smooth conjugation method, but made it possible to reach up to order eight in expansion orders. As with the previous formulations, it retained the periodic orbit structure, thus inheriting valuable information about the dynamics.

The corrections to the escape rate appeared to be a divergent series in the noise expansion parameter. They actually followed, via the cumulant expansion, from the traces of the evolution operator $\mathcal{L}$ [6], for which the noise correction series looked similarly divergent. In [7] the high order corrections were then worked out analytically for the special case of the first trace, $\mathrm{Tr}(\mathcal{L})$, as contour integrals asymptotically evaluated by the method of steepest descent. Here we further confirm the divergent nature of the high order noise corrections for $\mathrm{Tr}\mathcal{L}^n$; their high order terms are also controlled by sub-dominant saddles, which can be interpreted as generalized periodic orbits of some associated discrete Newtonian equations of motion.

In the following sections, first we define the stochastic dynamics, the evolution operator and its spectrum. Since this work was inspired by the results obtained for the eigenvalue corrections via the matrix representation method outlined in [6], for the sake of completeness we next show how to obtain a matrix representation of the evolution operator as an expansion in terms of the noise strength σ, and how to calculate the spectrum of the operator. Then we turn towards investigating the behavior of the high order noise corrections to the traces of $\mathcal{L}^n$, which are responsible for the divergent behavior seen at late terms of the eigenvalue corrections. Our key result is (59) where the high order noise corrections are converted into a trace formula. We give as a numerical example the quartic map considered in [4–7].

II. THE STOCHASTIC EVOLUTION OPERATOR

In this section we introduce the noisy repeller and its evolution operator. An individual trajectory in presence of additive noise is generated by iterating

$$x_{n+1} = f(x_n) + \sigma\xi_n \tag{1}$$

where $f(x)$ is a map, ξ_n a random variable with the normalized distribution $P(\xi)$, and σ parameterizes the noise strength. In what follows we shall assume that the mapping $f(x)$ is one-dimensional and expanding, and that the ξ_n are uncorrelated. A density of trajectories $\phi(x)$ evolves with time on the average as

$$\phi_{n+1}(y) = (\mathcal{L} \circ \phi_n)(y) = \int dx\, \mathcal{L}(y,x)\phi_n(x) \tag{2}$$

where the evolution operator $\mathcal{L}$ has the general form

$$\mathcal{L}(x,y) = \delta_\sigma(y - f(x)), \tag{3}$$

$$\delta_\sigma(x) = \int \delta(x - \sigma\xi)P(\xi)d\xi \;=\; \frac{1}{\sigma}P\left(\frac{x}{\sigma}\right). \tag{4}$$

For the calculations in this paper, Gaussian weak noise is assumed. The evolution operator

$$\mathcal{L}(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-f(x))^2}{2\sigma^2}} \tag{5}$$

can be expanded, in the perturbative limit $\sigma \to 0$, as

$$\mathcal{L}(x,y) \sim \sum_{N=0}^{\infty} a_{2N}\sigma^{2N}\delta^{(2N)}(y - f(x)), \tag{6}$$

where $\delta^{(2N)}$ denotes the $2N$-th derivative of the delta distribution and $a_{2N} = (2^N N!)^{-1}$ (or $a_{2N} = \frac{m_{2N}}{(2N)!}$ for general noise with finite n-th moment m_n). The map used for concrete calculations is the same as in our previous papers, a quartic map on the $(0,1)$ interval given by

$$f(x) = 20\left[\frac{1}{16} - \left(\frac{1}{2} - x\right)^4\right], \tag{7}$$

which is also shown on Fig. (1).

Throughout the theory developed in previous works [4–7], the periodic orbits of the system played a major role. A periodic orbit of length n was defined simply by

$$x_{j+1} = f(x_j), \qquad j = 1, ..., n \tag{8}$$

$$x_{n+1} \equiv x_1. \tag{9}$$

(This subscript j will quite generally be defined mod n.)

For a repeller the leading eigenvalue of the evolution operator yields a physically measurable property of the dynamical system, the escape rate from the repeller. In the case of deterministic flows, the periodic orbit theory yields explicit formulas for the spectrum of $\mathcal{L}$ as zeros of its spectral determinant [3]. Our goal here is to explore the dependence of the eigenvalues ν of $\mathcal{L}$ on the noise strength parameter σ.

The eigenvalues are determined by the eigenvalue condition

$$F(\sigma, \nu(\sigma)) = \det(1 - \mathcal{L}/\nu(\sigma)) = 0 \tag{10}$$

where $F(\sigma, 1/z) = \det(1 - z\mathcal{L})$ is the spectral determinant of the evolution operator $\mathcal{L}$. Computation of such determinants starts with evaluation of the traces of powers of the evolution operator

$$\mathrm{Tr}\frac{z\mathcal{L}}{1 - z\mathcal{L}} = \sum_{n=1}^{\infty} C_n z^n \,, \qquad C_n = \mathrm{Tr}\mathcal{L}^n \,, \tag{11}$$

which are then used to compute the cumulants $Q_n = Q_n(\mathcal{L})$ in the cumulant expansion

$$\det(1 - z\mathcal{L}) = 1 - \sum_{n=1}^{\infty} Q_n z^n \,, \tag{12}$$

by means of the recursion formula

$$Q_n = \frac{1}{n}\left(C_n - C_{n-1}Q_1 - \cdots C_1 Q_{n-1}\right) \tag{13}$$

which follows from the identity

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{L}^n\right) \,. \tag{14}$$

In the next section we show how to compute the cumulants Q_n using a local matrix representation of the evolution operator.

III. THE SPECTRUM OF THE EVOLUTION OPERATOR

As the mapping $f(x)$ is expanding by assumption, the evolution operator (2) smoothes the initial distribution $\phi(x)$. Hence it is natural to assume that the distribution $\phi_n(x)$ is analytic, and represent it as a Taylor series, intuition being that the action of $\mathcal{L}$ will smooth out fine detail in initial distributions and the expansion of $\phi_n(x)$ will be dominated by the leading terms in the series.

An analytic function $g(x)$ has a Taylor series expansion

$$g(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{\partial^m}{\partial y^m} g(y) \bigg|_{y=0} .$$

Following H.H. Rugh [12] we now define the matrix $(m, m' = 0, 1, 2, ...)$

$$(\mathbf{L})_{m'm} = \frac{\partial^{m'}}{\partial y^{m'}} \int dx \, \mathcal{L}(y,x) \frac{x^m}{m!} \bigg|_{y=0} . \tag{15}$$

$\mathbf{L}$ is a matrix representation of $\mathcal{L}$ which maps the x^m component of the density of trajectories $\phi_n(x)$ in (2) to the $y^{m'}$ component of the density $\phi_{n+1}(y)$, with $y = f(x)$. The desired traces can now be evaluated as traces of the matrix representation $\mathbf{L}$, $\mathrm{Tr}\mathcal{L}^n = \mathrm{Tr}\mathbf{L}^n$. As $\mathbf{L}$ is infinite dimensional, in actual computations we have to truncate it to a given finite order. For expanding flows the structure of $\mathbf{L}$ is such that its finite truncations give very accurate spectra.

Our next task is to evaluate the matrix elements of $\mathbf{L}$. Traces of powers of the evolution operator $\mathcal{L}^n$ are now also a power series in σ, with contributions composed of $\delta^{(m)}(f(x_a) - x_{a+1})$ segments. The contribution is non-vanishing only if the sequence $x_1, x_2, ..., x_n, x_{n+1} = x_1$ is a periodic orbit of the deterministic map $f(x)$. Thus the series expansion of $\mathrm{Tr}\mathcal{L}^n$ has support on all periodic points $x_a = x_{a+n}$ of period n, $f^n(x_a) = x_a$; the skeleton of periodic points of the deterministic problem also serves to describe the weakly stochastic flows. The contribution of the x_a neighborhood is best presented by introducing a coordinate system ϕ_a centered on the cycle points, and the operator (3) centered on the a-th cycle point

$$x_a \to x_a + \phi_a , \qquad a = 1, ..., n_p \tag{16}$$

$$f_a(\phi) \;=\; f(x_a + \phi) \tag{17}$$

$$\mathcal{L}_a(\phi_{a+1}, \phi_a) \;=\; \mathcal{L}(x_{a+1} + \phi_{a+1}, x_a + \phi_a)\,. \tag{18}$$

The weak noise expansion for the a-th segment operator is given by

$$\mathcal{L}_a(\phi', \phi) = \sum_{N=0}^{\infty} (-\sigma)^{2N} a_{2N} \delta^{(2N)}(\phi' + x_{a+1} - f_a(\phi))\,. \tag{19}$$

Repeating the steps that led to (15) we construct the local matrix representation of $\mathcal{L}_a$ centered on the $x_a \to x_{a+1}$ segment of the deterministic trajectory

$$(\mathbf{L}a)_{m'm} \;=\; \frac{\partial^{m'}}{\partial \phi'^{m'}} \int d\phi\, \mathcal{L}_a(\phi', \phi) \frac{\phi^m}{m!}\bigg|_{\phi'=0}\,. \tag{20}$$

$$\;=\; \sum_{N=max(m-m',0)}^{\infty} (-\sigma)^{2N} a_{2N} (\mathbf{B}a)_{m'+N,m}\,. \tag{21}$$

Due to its simple dependence on the Dirac delta function, $\mathbf{B}a$ can expressed in terms of derivatives of the inverse of $f_a(\phi)$:

$$(\mathbf{B}a)_{nm} = \frac{\partial^n}{\partial \phi'^n} \int d\phi\, \delta(\phi' + x_{a+1} - f_a(\phi)) \frac{\phi^m}{m!}\bigg|_{\phi'=0} \tag{22}$$

$$= \frac{\partial^n}{\partial \phi'^n} \frac{(f_a^{-1}(x_{a+1} + \phi') - x_a)^m}{m! |f_a'(f_a^{-1}(x_{a+1} + \phi'))|}\bigg|_{\phi'=0} \tag{23}$$

$$= \frac{\mathrm{sign}(f_a')}{(m+1)!} \; \frac{\partial^{n+1}(\mathcal{F}_a(\phi')^{m+1})}{\partial \phi'^{n+1}}\bigg|_{\phi'=0}\,, \tag{24}$$

where we introduced the shorthand notation $\mathcal{F}_a(\phi') \;=\; f_a^{-1}(x_{a+1} + \phi') - x_a$. The matrix elements can be easily worked out explicitly using (24). In [6] we show that $\mathbf{B}a$ is a lower triangular matrix, in which the diagonal terms drop off exponentially, and the terms below the diagonal fall off even faster. This way truncating $\mathbf{B}a$ is justified, as truncating the matrix to a finite one introduces only exponentially small errors.

In the local matrix approximation the traces of evolution operator are approximated by

$$\mathrm{Tr}\mathcal{L}^n\big|_{\mathrm{saddles}} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p r} \mathrm{Tr}\mathbf{L}p^r = \sum_{N=0}^{\infty} C_{n,N} \sigma^{2N}\,, \tag{25}$$

where $\mathrm{Tr}\mathbf{L}p = \mathrm{Tr}\mathbf{L}_{n_p}\mathbf{L}_2 \cdots \mathbf{L}_1$ is the contribution of the p cycle, and the power series in σ^{2N} follows from the expansion (21) of $\mathbf{L}a$ in terms of $\mathbf{B}a$. The traces of $\mathbf{L}^n$ evaluated by (21) yield a series in σ^{2N}, and the σ^{2N} coefficients $Q_{n,N}$ in the cumulant expansion

$$F = \det(1 - z\mathcal{L}) = 1 - \sum_{n=1}^{\infty}\sum_{N=0}^{\infty} Q_{n,N} z^n \sigma^{2N} \tag{26}$$

are then obtained recursively from the traces, as in (13):

$$Q_{n,N} = \frac{1}{n}\left(C_{n,N} - \sum_{k=1}^{n-1}\sum_{l=0}^{N} Q_{k,N-l} C_{n-k,l} \right) . \tag{27}$$

From the cumulants, by manipulating formal Taylor series, we can calculate perturbative corrections to the eigenstates.

Figure (2) shows the cumulants obtained in the numerical tests and table (I) shows the eigenvalue corrections computed from the cumulants. Both the cumulants and the eigenvalue corrections exhibit a super-exponential convergence with the truncation cycle length n, whereas the corrections appear to form a divergent series in the noise parameter σ.

IV. TRACE FORMULA FOR NOISE CORRECTIONS

With as many as eight orders of perturbation theory, we should now turn towards investigating the asymptotic nature of these perturbative expansions. The corrections to the escape rate are calculated via the cumulant expansion from the traces of the evolution operators $\mathcal{L}^n$. These traces are responsible for the asymptotic behavior since they themselves apparently have divergent high order noise corrections. Thus in this section we investigate the behavior of the late terms in the noise expansion series of the traces $\mathrm{Tr}(\mathcal{L}^n)$. With the help of an integral representation of the operator we shall be able to transform $\mathrm{Tr}(\mathcal{L}^n)$ to contour integrals, and then we will evaluate these integrals in the saddle point approximation to arrive to a formula analogous to the usual trace formulas arising in quantum chaos.

The trace of $\mathcal{L}^n$ can be expressed as

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int dx_1 dx_2 ... dx_n e^{-\frac{S(\vec{x})}{\sigma^2}}, \tag{28}$$

where

$$S(\vec{x}) = \frac{1}{2}\sum_{j=1}^{n} \left(x_{j+1} - f(x_j)\right)^2, \tag{29}$$

$$\vec{x} = (x_1, \cdots, x_n), \qquad \text{with} \quad x_{n+1} \equiv x_1. \tag{30}$$

In order to give a deeper insight on the forthcoming calculations, we draw a correspondence between our system and a discrete Hamiltonian mechanics, with the S defined above playing the role of the classical action. According to (29), the least action principle requires

$$x_j - f(x_{j-1}) - f'(x_j)(x_{j+1} - f(x_j)) = 0. \tag{31}$$

We define

$$\vec{p} = (p_1, \cdots, p_n), \qquad p_j := x_j - f(x_{j-1}), \tag{32}$$

the quantity corresponding to the momentum in the classical mechanics. From (31) we obtain

$$x_{j+1} = f(x_j) + p_{j+1}, \tag{33}$$

$$p_{j+1} = \frac{p_j}{f'(x_j)}, \tag{34}$$

which are the equations corresponding to the classical Newtonian equations of motion. The generalized periodic orbits of length n are those orbits, which obey these equations and $x_{n+1} = x_1$, $p_{n+1} = p_n$. Those generalized periodic orbits which have non-zero momentum will control the asymptotic behavior of the corrections to $\mathrm{Tr}\mathcal{L}^n$ as we shall demonstrate later. The original periodic orbits defined by (8),(9) are those with zero momentum. The generalized periodic orbits with non-zero momentum and the original periodic orbits proliferate with growing n as suggested by Fig 3.

We introduce an integral representation of the noisy kernel, which will be of great use in the later calculations:

$$\mathcal{L}(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-f(x))^2}{2\sigma^2}} = \frac{1}{2\pi}\int dk e^{-\frac{\sigma^2 k^2}{2} + ik(y-f(x))}. \tag{35}$$

Using this new integral representation,

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(2\pi)^n}\int dk^n dx^n e^{-\frac{\sigma^2}{2}\sum_{j=1}^n k_j^2 + i\sum_{j=1}^n k_j(x_{j+1}-f(x_j))}, \tag{36}$$

or equivalently

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(2\pi)^n}\int dk^n \int dp^n J_n(\vec{p})\, e^{-\frac{\sigma^2}{2}\sum_{j=1}^n k_j^2 + i\sum_{j=1}^n k_j p_j}, \tag{37}$$

where $J_n(\vec{p})$ denotes the Jacobian $D(\vec{x})/D(\vec{p})$. Since

$$\frac{1}{(2\pi)^n}\int dk^n e^{i\sum_{j=1}^n k_j p_j} = \prod_{j=1}^n \delta(p_j), \tag{38}$$

we can reduce (37) to

$$\mathrm{Tr}\mathcal{L}^n = \int dp^n J_n(\vec{p})\, e^{\frac{\sigma^2}{2}\Delta_n}\prod_{j=1}^n \delta(p_j) = e^{\frac{\sigma^2}{2}\Delta_n} J_n(\vec{p})\Big|_{p_j=0}, \tag{39}$$

with Δ_n denoting the Laplacian

$$\Delta_n = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + ... + \frac{\partial^2}{\partial p_n^2}. \tag{40}$$

We focus on the Taylor expansion of (39) in the noise parameter:

$$\mathrm{Tr}\mathcal{L}^n = \sum_{N=0}^{\infty} \left(\mathrm{Tr}\mathcal{L}^n\right)_N \sigma^{2N}, \tag{41}$$

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N = \frac{1}{2^N}\frac{(\Delta_n)^N}{N!} J_n(\vec{p})\Big|_{p_j=0}. \tag{42}$$

The N-th power of the Laplacian in the equation above can be written as

$$(\Delta_n)^N = \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{N!}{j_1!\ldots j_n!}\frac{\partial^{2j_1}}{\partial p_1^{2j_1}}\cdots\frac{\partial^{2j_n}}{\partial p_n^{2j_n}}\delta_{N,\sum_{k=1}^n j_k}, \tag{43}$$

where δ_{jl} is the Kronecker-delta (making the sum actually finite). With the help of the multidimensional residue formula from complex calculus [13]

$$\frac{\partial^{n_1+\ldots+n_k} f\left(\vec{z}\right)}{\partial z_1^{n_1}\ldots\partial z_k^{n_k}} =$$
$$\frac{n_1!\ldots n_k!}{(2\pi i)^k}\oint\ldots\oint\frac{f(\vec{\xi})d\xi_1\ldots d\xi_k}{(\xi_1-z_1)^{n_1+1}\ldots(\xi_k-z_k)^{n_k+1}}, \tag{44}$$

we obtain

$$(\mathrm{Tr}\mathcal{L}^n)_N = \frac{1}{(2\pi i)^n 2^N}\sum_{j_1,\ldots,j_n=0}^{\infty}\frac{(2j_1)!}{j_1!}\ldots\frac{(2j_n))!}{j_n!}$$
$$\times\delta_{N,\sum_{k=1}^n j_k}\oint_{c_1}\ldots\oint_{c_n}\frac{J_n\left(\vec{p}\right)dp_1\ldots dp_n}{p_1^{2j_1+1}\ldots p_n^{2j_n+1}}. \tag{45}$$

These contours encircle the $p_j=0$ points positively. The integrals can be transformed back to contour integrals in the original x_j variables, and the contours will be placed around the original periodic orbits of the system defined by (8–9), since it is these orbits which fulfill the $p_j=0$ conditions.

$$(\mathrm{Tr}\mathcal{L}^n)_N = \frac{1}{(2\pi i)^n 2^N}\sum_{j_1,\ldots,j_n=0}^{\infty}\frac{(2j_1)!}{j_1!}\ldots\frac{(2j_n))!}{j_n!}$$
$$\times\delta_{N,\sum_{k=1}^n j_k}\oint_{C_1}\ldots\oint_{C_n}$$
$$\frac{dx_1\ldots dx_n}{(x_1-f(x_n))^{2j_1+1}\ldots(x_n-f(x_{n-1}))^{2j_n+1}}. \tag{46}$$

From now on we shall restrict our calculations to the asymptotic large-N limit. We will replace the summations in (45) by integrals and then use the saddle-point method to get a compact formula for $(\mathrm{Tr}\mathcal{L}^n)_N$. Then we may consistently approximate the factorials via the Stirling formula [14] as

$$\frac{(2j_k)!}{j_k!}\simeq\frac{\left(\frac{2j_k}{e}\right)^{2j_k}\sqrt{4\pi j_k}}{\left(\frac{j_k}{e}\right)^{j_k}\sqrt{2\pi j_k}} = 2^{2j_k+1/2}j_k^{j_k}e^{-j_k} =$$
$$2^{1/2}e^{2(\ln 2)j_k+j_k\ln j_k-j_k}. \tag{47}$$

Using (47) and an integral representation of the delta function we get

$$(\mathrm{Tr}\mathcal{L}^n)_N\simeq\frac{2^{\frac{n}{2}-N}}{(2\pi i)^n 2\pi}\sum_{j_1,\ldots,j_n=0}^{\infty}\int dt\oint_{C_1}\ldots\oint_{C_n}dx_1\ldots dx_n$$
$$\times\exp\left[it(N-\sum_{k=1}^n j_k)+(2\ln 2-1)\sum_{k=1}^n j_k+\sum_{k=1}^n j_k\ln j_k\right.$$
$$\left.+\sum_{k=1}^n\ln(x_k-f(x_{k-1}))(2j_k+1)\right]. \tag{48}$$

Now we replace j_k with the new variables $y_k = \frac{j_k}{N}$ and in the asymptotic (N large) limit approximate the summations over j_k with integrals over y_k as

$$\begin{aligned}
&(\mathrm{Tr}\mathcal{L}^n)_N \simeq \\
&\frac{2^{\frac{n}{2}-N}N^n}{(2\pi i)^n 2\pi}\int_0^\infty dy_1 ... \int_0^\infty dy_n \int dt \oint_{C_1} ... \oint_{C_n} dx_1...dx_n \\
&\times \exp\Bigg[it(N - N\sum_{k=1}^n y_k) + N(2\ln 2 - 1)\sum_{k=1}^n y_k \\
&+N\sum_{k=1}^n y_k \ln(Ny_k) + \sum_{k=1}^n \ln(x_k - f(x_{k-1}))(2Ny_k + 1)\Bigg].
\end{aligned} \tag{49}$$

We evaluate the y integrals with the saddle-point method to get

$$\begin{aligned}
(\mathrm{Tr}\mathcal{L}^n)_N \simeq & \frac{2^{-N+\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} i^n 2\pi}\int dt \oint_{C_1} ... \oint_{C_n} dx_1...dx_n \\
& \exp\left[it\left(N + \frac{n}{2}\right) - e^{it}\frac{S}{2}\right].
\end{aligned} \tag{50}$$

Next we implement the saddle-point method to the integral in t as well, asymptotically resulting in

$$\begin{aligned}
(\mathrm{Tr}\mathcal{L}^n)_N \simeq & \frac{N^{\frac{n-1}{2}}}{2^{2N+\frac{1}{2}}(2\pi)^{\frac{n+1}{2}} i^{n+1}}\frac{(2N)!}{N!} \\
& \times \int dx^n e^{-\left(N+\frac{n}{2}\right)\ln(S)}.
\end{aligned} \tag{51}$$

The last step is to evaluate the contour integrals in the x_k variables. We deform the contours until the saddle-points are reached and the contours run along the paths of steepest descent. The leading contribution comes from the saddle-points, which fulfill the following equation

$$\frac{1}{S}\left[x_j^* - f(x_{j-1}^*) - (x_{j+1}^* - f(x_j^*))f'(x_j^*)\right] = 0. \tag{52}$$

By comparing (52) and (31) one can see that the saddle-points are all generalized periodic orbits of the system. Since the contours ran originally around the orbits with zero momentum, these do not come into account as saddle-points. The second derivative matrix is

$$-\left(N+\frac{n}{2}\right)\frac{1}{S}D^2S, \tag{53}$$

where D^2S denotes the second derivative matrix of S

$$(D^2S)_{ij} = \frac{\partial^2 S}{\partial x_i \partial x_j}. \tag{54}$$

This would be the matrix to deal with if we were taking the saddle-point approximation of (28) directly. We reorganize the prefactor in (51) with the use of the Stirling formula [14]; then the result of the saddle-point integration comes out as

$$\begin{aligned}&(\mathrm{Tr}\mathcal{L}^n)_N \simeq \\ &\sum_{\text{s.p.}} \frac{N^{\frac{n-1}{2}}}{2\pi i}\frac{\Gamma(N+\frac{1}{2})}{\left(N+\frac{n}{2}\right)^{\frac{n}{2}}}\frac{S_p^{-N}}{\sqrt{\det D^2S_p}}.\end{aligned} \tag{55}$$

For $n=1$ this formula restores the result of [7] as it should.

Finally we draw attention to the close connection between the generalized periodic orbits of the system and D^2S. The stability matrix of a generalized periodic orbit p is expressed as

$$M_p = M(n)\cdot M(n-1)\cdot M(n-2)\cdots M(1) \tag{56}$$

$$M(k) = \begin{pmatrix} f'(x_k) - \frac{p_k}{(f'(x_k))^2}f''(x_k) & \frac{1}{f'(x_k)} \\ -\frac{p_k}{(f'(x_k))^2}f''(x_k) & \frac{1}{f'(x_k)} \end{pmatrix} \tag{57}$$

The determinant of D^2S can be expressed with the help of the stability matrix as

$$\det D^2S_p = \det(M_p - 1). \tag{58}$$

This way we reformulate (55) as

$$(\mathrm{Tr}\mathcal{L}^n)_N \simeq \frac{N^{\frac{n-1}{2}}}{2\pi}\frac{\Gamma(N+\frac{1}{2})}{\left(N+\frac{n}{2}\right)^{\frac{n}{2}}}\sum_{\text{g.p.o.}}\frac{e^{-N\ln S_p}}{\sqrt{\det(1-M_p)}}, \tag{59}$$

where the summation runs over generalized periodic orbits, with non-zero momentum. This expression, fully analogous to a trace formula, is our main result.

Finally, we provide a numerical test of our high order estimates. In [7] we developed a contour integral method to calculate high order noise corrections to the trace of $\mathcal{L}$, and

obtained a very good agreement between the exact results and a formula which is just the approximation of (55) in the $n = 1$ case. Now we can similarly verify the high order noise corrections to the trace of $\mathcal{L}^2$. Fig. 4 plots, as a function of N, the ratios of the approximations for $(\mathrm{Tr}\mathcal{L})_N$ and $(\mathrm{Tr}\mathcal{L}^2)_N$ obtained from (55) to the exact results (46) computed using direct numerical evaluations of the contour integrals therein. (For higher powers of $\mathcal{L}$, the proliferation of the orbits forces the contours to run close to singular points, making efficiently precise numerical integrations extremely time-consuming).

In summary, we have studied the evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and a Gaussian noise distribution. Motivated by the divergent growth of the high order eigenvalue corrections of the evolution operator, we investigated the behavior of the high order noise corrections to the traces of $\mathcal{L}^n$. Using an integral representation of the evolution operator $\mathcal{L}$, we found that the asymptotic behavior of the corrections to the trace of $\mathcal{L}^n$ is governed by sub-dominant quantities previously neglected in the perturbative expansion, and a full-fledged trace formula can be derived for the late terms in the noise expansion series of the trace of $\mathcal{L}^n$.

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FIGURES

FIG. 1. The map (7) on the [0,1] interval.

FIG. 2. The perturbative corrections to the cumulants $Q_{n,N}$ plotted as a function of cycle length n (for perturbation orders $N = 0, 2, 4, 6, 8$); all exhibit super-exponential convergence in n.

FIG. 3. The shortest original and generalized periodic orbits of the map (7). Squares mark original periodic points, dots mark generalized periodic points. Large symbols indicate orbits of length one, small symbols indicate orbits of length two.

FIG. 4. The ratios of $(\mathrm{Tr}\mathcal{L})_N$ and $(\mathrm{Tr}\mathcal{L}^2)_N$ as calculated via the asymptotic formula (55) to their values computed by numerical integration of formula (46).

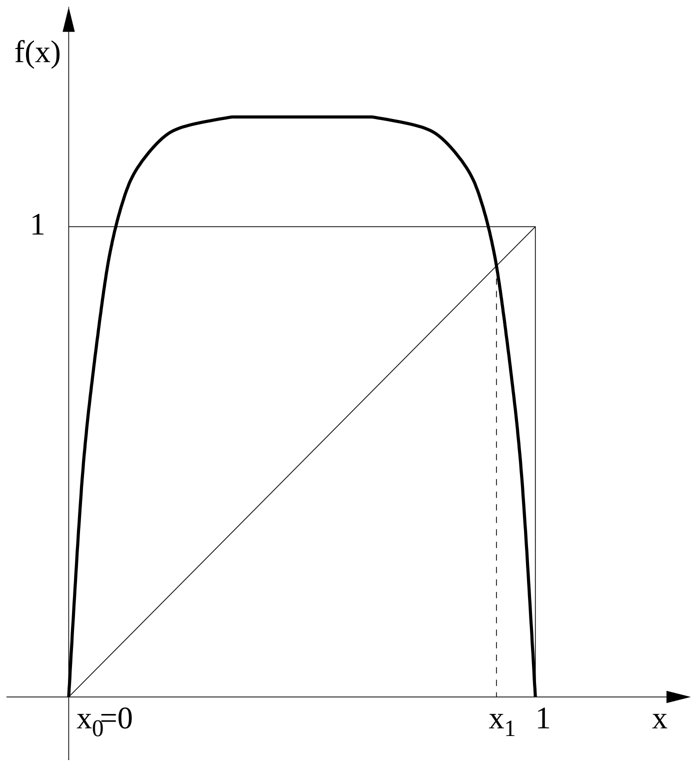
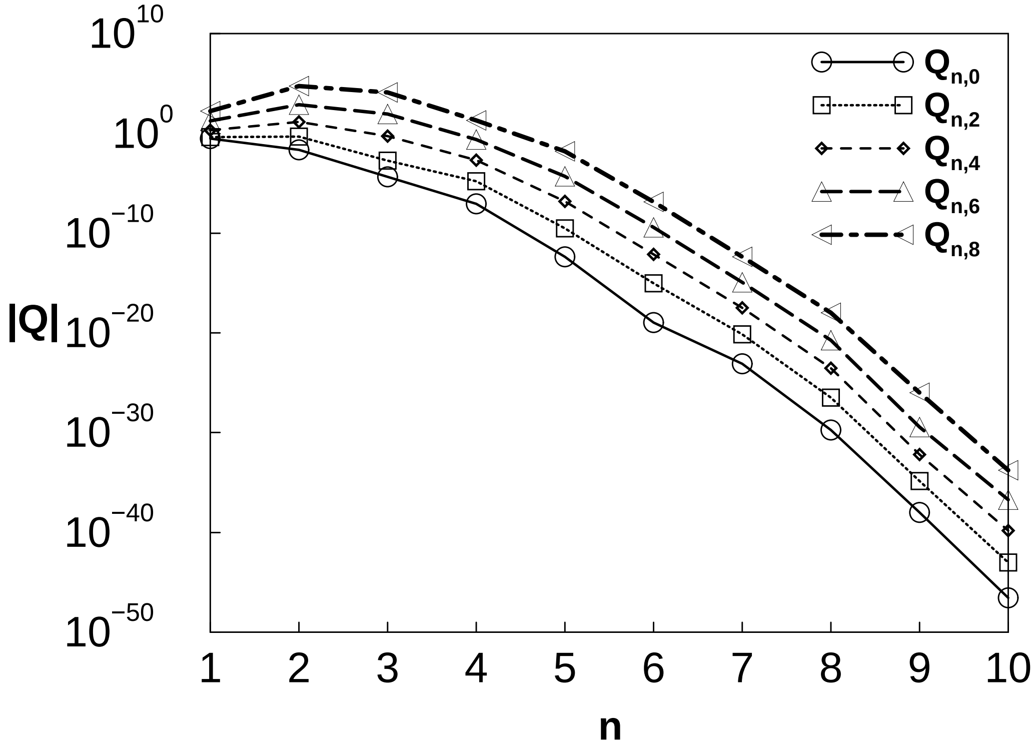


FIG.1.



10^{10}
10^{0}
10^{-10}
10^{-20}
10^{-30}
10^{-40}
10^{-50}
|Q|
1
2
3
4
5
6
7
8
9
10
n
$Q_{n,0}$
$Q_{n,2}$
$Q_{n,4}$
$Q_{n,6}$
$Q_{n,8}$

FIG.2.

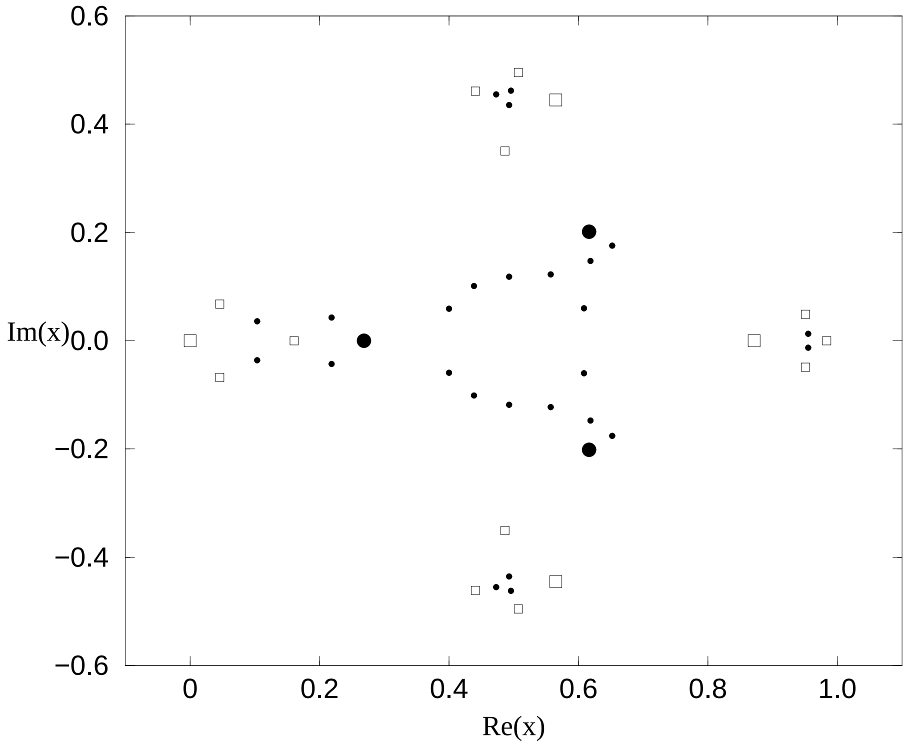


FIG.3.

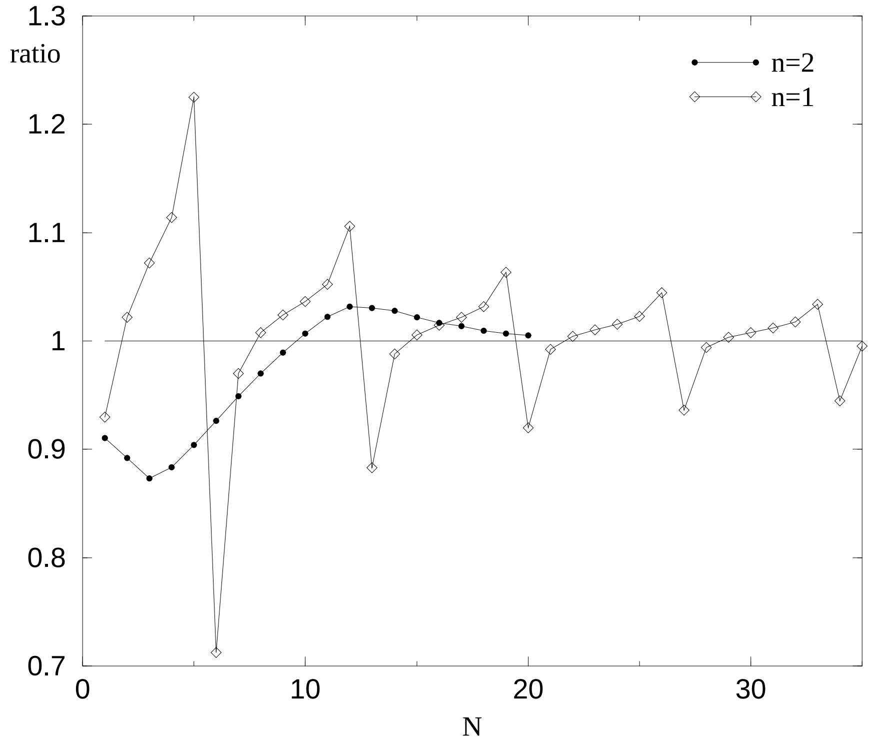


FIG.4.

TABLES

n	ν_0	ν_2	ν_4	ν_6	ν_8
1	0.308	0.42	2.2	17.4	168.0
2	0.37140	1.422	32.97	1573.3	112699.9
3	0.3711096	1.43555	36.326	2072.9	189029.0
4	0.371110995255	1.435811262	36.3583777	2076.479	189298.8
5	0.371110995234863	1.43581124819737	36.35837123374	2076.4770492	189298.12802
6	0.371110995234863	1.43581124819749	36.358371233836	2076.47704933320	189298.128042526

TABLE I. Significant digits of the leading deterministic eigenvalue ν_0, and of its $\sigma^2, \cdots, \sigma^8$ perturbative coefficients $\nu_2, \cdots, \nu_8$, calculated from the cumulant expansion of the spectral determinant as a function of the cycle truncation length n. Note the super-exponential convergence of all coefficients.

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